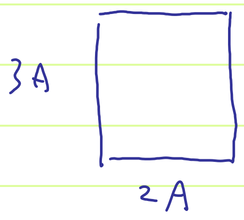
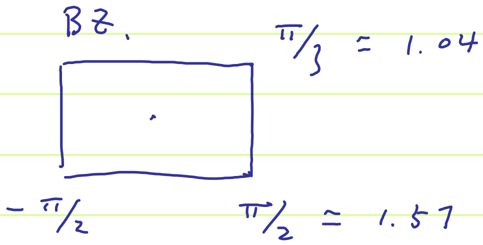
SSP Final Exam

1. (a) real space $\qquad$ k space



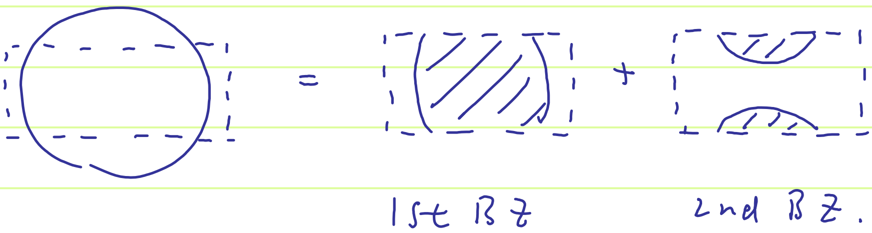


2-dim

$$2 \cdot \frac{\pi k_F^2}{\left(\frac{2\pi}{ab}\right)^2} = N \rightarrow k_F = \sqrt{2\pi n}$$

$$n = \frac{2}{2\cdot 3} = \frac{1}{3}\frac{1}{A^2}, \quad k_F = \sqrt{\frac{2\pi}{3}} \simeq 1.4 \frac{1}{A}$$

(b)



2 (a)

$$H = t\sum_{j=0}^{2N-1} |j\rangle\langle j+1| + |j+1\rangle\langle j| + h\sum_j (-1)^j |j\rangle\langle j|$$

$$= t\sum_{j=0}^{N-1}\Big(|a_j\rangle\langle b_j| + |b_j\rangle\langle a_j| + h\sum_{j=0}^{N-1}\big(|a_j\rangle\langle a_j|$$

$$+ |b_j\rangle\langle a_{j+1}| + |a_{j+1}\rangle\langle b_j|\Big) \qquad - |b_j\rangle\langle b_j|\big)$$

$a_N = a_0, \ b_N = b_0$

$$\begin{cases} |a_j\rangle = \frac{1}{\sqrt{N}}\sum_k e^{ik2j}|a_k\rangle \\ |b_j\rangle = \frac{1}{\sqrt{N}}\sum_k e^{ik(2j+1)}|b_k\rangle \end{cases}$$

$$\rightarrow \quad H = t \frac{1}{N} \sum_{k,k'} \sum_{j} e^{ikz_j} e^{-ik'(z_j+1)} |a_k\rangle\langle b_{k'}| + h.c.$$

$$+ t \frac{1}{N} \sum_{k,k'} \sum_{j} e^{ik(z_j+1)} e^{-ik'(z_j+2)} |b_k\rangle\langle a_{k'}| + h.c.$$

$$+ h \frac{1}{N} \sum_{k,k'} \sum_{j} e^{ikz_j} e^{-ik'z_j} \left(|a_k\rangle\langle a_{k'}| - |b_k\rangle\langle b_{k'}| \right)$$

use $\frac{1}{N} \sum_j e^{i(k-k')z_j} = \delta_{kk'}$

$$\rightarrow \quad H = t \sum_k \left(e^{-ik} |a_k\rangle\langle b_k| + e^{ik} |b_k\rangle\langle a_k| \right)$$

$$+ t \sum_k \left(e^{-ik} |b_k\rangle\langle a_k| + e^{ik} |a_k\rangle\langle b_k| \right)$$

$$+ h \sum_k \left(|a_k\rangle\langle a_k| - |b_k\rangle\langle b_k| \right)$$

$$= 2t \sum_k \cos k \left(|a_k\rangle\langle b_k| + |b_k\rangle\langle a_k| \right)$$

$$+ h \sum_k \left(|a_k\rangle\langle a_k| - |b_k\rangle\langle b_k| \right)$$

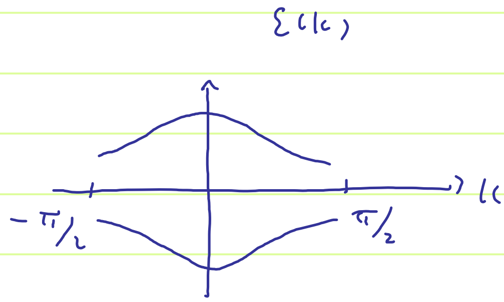
(b)

$$= \sum_k \left(|a_k\rangle, |b_k\rangle \right) \begin{pmatrix} h & 2t\cos k \\ 2t\cos k & -h \end{pmatrix} \begin{pmatrix} \langle a_k| \\ \langle b_k| \end{pmatrix}$$

$$\det \begin{pmatrix} h-\lambda & 2t\cos k \\ 2t\cos k & -h-\lambda \end{pmatrix} = 0$$

$$\rightarrow \quad \lambda^2 - h^2 - (2t\cos k)^2 = 0$$

$$\rightarrow \quad \lambda = \pm\sqrt{(2t\cos k)^2 + h^2}$$



3. (a) total kinetic energy

$$T = 2 \sum_k \frac{\hbar^2 k^2}{2m}$$

$$= 2V \int \frac{d^2k}{(2\pi)^2} \frac{\hbar^2 k^2}{2m}$$

$$= \frac{V\hbar^2}{m} \underbrace{\int_0^{k_F} \frac{k\,dk}{2\pi} k^2}_{\frac{1}{8\pi} k_F^4} = \frac{V\hbar^2}{8\pi m} k_F^4$$

$$\frac{T}{N} = \frac{\hbar^2}{8\pi m} \frac{k_F^4}{n} = \frac{\hbar^2 k_F^2}{4m} \quad \leftarrow \quad k_F^2 = 2\pi n$$

(b) in the jellium approx, Hartree energy $= 0$

Fock energy:

1st, $g_{ss'}(r, r') = 1 - \left| \frac{2}{N} \sum_k e^{i\bar{k}\cdot(\bar{r}-\bar{r}')} \right| \delta_{ss'}$

$$\frac{2}{N} \sum_k e^{i\bar{k}\cdot\bar{r}} = \frac{2V}{N} \int \frac{d^2k}{(2\pi)^2} e^{ikr\cos\phi}$$

$$= \frac{2}{n} \frac{1}{4\pi^2} \int_0^{k_F} k\,dk \underbrace{\int_0^{2\pi} d\phi\, e^{ikr\cos\phi}}_{2\pi J_0(kr)}$$

use $\int_0^{k_F} J_0(kr)\, k\,dk = \frac{1}{r^2} \int_0^{x_F} J_0(x)\, x\,dx$

$$= \frac{1}{r^2} (k_F r) J_1(k_F r)$$

$$= \frac{2\pi}{2\pi^2 n} \frac{k_F^2}{k_F r} J_1(k_F r)$$

$$= 2 \frac{J_1(k_F r)}{k_F r} \quad (k_F^2 = 2\pi n)$$

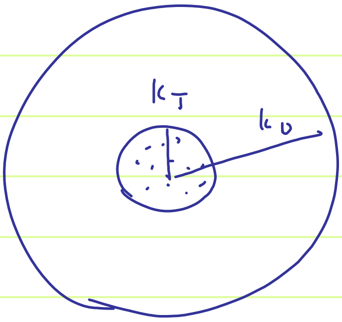
note: $\mathcal{E}_x = -e \int d^2r \frac{en(1-g_{\uparrow\uparrow})}{r} \frac{1}{2}$ (see lecture note)

$$= -\frac{e^2 n}{2} \int d^2r \frac{1}{r} \left(\frac{2J_1(k_F r)}{k_F r} \right)^2$$

$$= -2e^2 n \frac{2\pi}{k_F} \int_0^{x_R} dx \frac{J_1^2(x)}{x^2} \quad x_R = k_F R$$

would diverge

4 (a) heuristic argument



fraction of excited modes

$$\sim (k_T/k_D)^2 = (T/\theta)^2$$

energy absorbed

$$U \sim kT\, N \left(\frac{T}{\theta}\right)^2 \propto T^3$$

$$\therefore \quad C = \frac{dU}{dT} \propto T^2$$

(b)

$$I = \frac{1}{N} \sum_{ks} \frac{n_{ks} + 1/2}{\hbar \omega_{ks}}$$

Debye model

$$= \frac{3}{N} \int d\omega\, D(\omega) \left(\langle n_k \rangle + \frac{1}{2} \right) \frac{1}{\hbar \omega} \quad , \quad D(\omega) = \frac{V \omega^2}{2\pi^2 v^3}$$

$$= \frac{3V}{2\pi^2 N v^3 \hbar} \underbrace{\int_0^{\omega_D} d\omega \left(\frac{\omega}{e^{\hbar\omega/kT} - 1} + \frac{\omega}{2} \right)}$$

$$= \frac{(kT)^2}{\hbar^2} \int_0^{x_D} dx \left(\underbrace{\frac{x}{e^x - 1} + \frac{x}{2}} \right) \quad , \quad x_D = \frac{\hbar \omega_D}{kT} = \frac{\theta}{T}$$

at $T \gg \theta$, $\quad \simeq 1 + \frac{x}{2}$

$$\simeq \left(\frac{kT}{\hbar} \right)^2 \left[\frac{\theta}{T} + \frac{1}{4} \left(\frac{\theta}{T} \right)^2 \right]$$

$$\simeq \frac{3V}{2\pi^2 N v^3 \hbar^3} k^2\, T\theta$$

$k_D^3 = 6\pi^2 n \longrightarrow$

$$= \frac{9}{(\underbrace{v\hbar k_D}_{\hbar\omega_D = k\theta})^3} k^2\, T\theta = \frac{9}{k\theta} \frac{T}{\theta}$$