Chap 5 Inhomogeneous superconductor

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I. BOGOLIUBOV-DE GENNE EQUATION

A. Pairing Hamiltonian in real space

In this chapter, we study superconductors that do not have a uniform electronic structure. This happens, for example, if the superconductor has vortices, a surface, or an interface with another material. Without translation symmetry, momentum is no longer a good quantum number. Therefore, we start with a pairing interaction with space variables,

$$H_{ee} = -V_e \int dV \psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r}). \tag{1}$$

Define the **pairing potential** as

$$\Delta(\mathbf{r}) = -V_e\langle\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle, \tag{2}$$

which is similar to the gap function defined in Chap 4. Then the quartic term can be decomposed as

$$-V_e\psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow \simeq \Delta(\mathbf{r})\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \Delta^*(\mathbf{r})\psi_\downarrow\psi_\uparrow. \tag{3}$$

The mean-field Hamiltonian becomes

$$H_{eff} = \int dV \left\{ \sum_s \psi_s^\dagger H_0 \psi_s + \Delta(\mathbf{r})\psi_\uparrow^\dagger\psi_\downarrow^\dagger + \Delta^*(\mathbf{r})\psi_\downarrow\psi_\uparrow \right\}, \tag{4}$$

where

$$H_0 = \frac{1}{2m}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2 + U(\mathbf{r}) - \mu, \tag{5}$$

in which $U(\mathbf{r})$ is an external potential.

We will use the eigenstates of H_0 without $\mathbf{A}$ as basis,

$$\left(\frac{p^2}{2m} + U - \mu\right) w_n = \varepsilon_n w_n. \tag{6}$$

Because of the Kramer degeneracy, each eigenvalue ε_n is two-fold degenerate: $w_n(\mathbf{r}) \otimes |\uparrow\rangle$ and $w_n^*(\mathbf{r}) \otimes |\downarrow\rangle$ have the same energy.

The BCS state is generalized as

$$|\psi_G\rangle = \prod_n \left(u_n + v_n c_{n\uparrow}^\dagger c_{n\downarrow}^\dagger\right)|\mathbf{0}\rangle, \tag{7}$$

where $c_{n\downarrow}^\dagger|\mathbf{0}\rangle$ is a time-reversed state of $c_{n\uparrow}^\dagger|\mathbf{0}\rangle$, which creates the state $w_n(\mathbf{r}) \otimes |\uparrow\rangle$.

B. Bogoliubov-de Gennes equation

The pairing Hamiltonian H_{eff} is not diagonal under the eigenstate basis of H_0. We wish to diagonalze H_{eff} using a generalized Bogoliubov-Valatin transformation (here we follow de Gennes' convention in Ref. 1 while writing the coefficients u_n and v_n),

$$\begin{aligned} \psi_\uparrow(\mathbf{r}) &= \sum_n \left(u_n(\mathbf{r})\gamma_{n\uparrow} - v_n^*(\mathbf{r})\gamma_{n\downarrow}^\dagger\right), \\ \psi_\downarrow(\mathbf{r}) &= \sum_n \left(v_n^*(\mathbf{r})\gamma_{n\uparrow}^\dagger + u_n(\mathbf{r})\gamma_{n\downarrow}\right). \end{aligned} \tag{8}$$

We have to choose u_n and v_n such that

$$H_{eff} = \sum_{ns} E_n \gamma_{ns}^\dagger\gamma_{ns}, \tag{9}$$

and

$$\begin{aligned} \{\gamma_{ns}, \gamma_{n's'}^\dagger\} &= \delta_{nn'}\delta_{ss'}, \\ \{\gamma_{ns}, \gamma_{n's'}\} &= 0. \end{aligned} \tag{10}$$

This leads to

$$\begin{aligned} i\dot{\gamma}_{ns} &= [\gamma_{ns}, H_{eff}] = E_n\gamma_{ns}, \\ i\dot{\gamma}_{ns}^\dagger &= \left[\gamma_{ns}^\dagger, H_{eff}\right] = -E_n\gamma_{ns}^\dagger. \end{aligned} \tag{11}$$

On the other hand, from Eq. (4), one has

$$\begin{aligned} i\dot{\psi}_\uparrow(\mathbf{r}) &= [\psi_\uparrow(\mathbf{r}), H_{eff}] = H_0\psi_\uparrow(\mathbf{r}) + \Delta(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r}), \\ i\dot{\psi}_\downarrow(\mathbf{r}) &= [\psi_\downarrow(\mathbf{r}), H_{eff}] = H_0\psi_\downarrow(\mathbf{r}) - \Delta(\mathbf{r})\psi_\uparrow^\dagger(\mathbf{r}). \end{aligned} \tag{12}$$

Rewrite ψ's using γ's and compare with Eq. (11). One can get

$$\begin{aligned} H_0u_n(\mathbf{r}) + \Delta(\mathbf{r})v_n(\mathbf{r}) &= E_nu_n(\mathbf{r}), \\ H_0^*v_n(\mathbf{r}) - \Delta^*(\mathbf{r})u_n(\mathbf{r}) &= -E_nv_n(\mathbf{r}). \end{aligned} \tag{13}$$

This is called the **Bogoliubov-de Gennes** (BdG) equations. They can be written in the following matrix form,

$$\begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0^* \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = E_n\begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{14}$$

A few remarks are in order:

First, if $(u_n, v_n)^T$ is a solution with energy E_n, then $(-v_n^*, u_n^*)$ is a solution with energy $-E_n$. But we only keep the positive excitation energy.

Second, for a uniform system,

$$u_n(\mathbf{r}) = u_k e^{i\mathbf{k}\cdot\mathbf{r}},$$
$$v_n(\mathbf{r}) = v_k e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (15)$$

In this case, the BdG equation immediately gives us the E_k and (u_k, v_k) in Chap 4.

Third, if $\Delta = 0$, then

$$H_0 u_n = \varepsilon_n u_n,$$
$$H_0 v_n = -\varepsilon_n v_n, \quad (16)$$

so that (u_n, v_n) are the electron and hole eigenfunctions for the normal state.

The pairing potential in the BdG equation is unknown,

$$\begin{aligned}\Delta(\mathbf{r}) &= V_e \langle \psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r})\rangle \\ &= V_e \sum_n u_n v_n^* \left(1 - \langle \gamma_{n\uparrow}^\dagger \gamma_{n\uparrow}\rangle - \langle \gamma_{n\downarrow}^\dagger \gamma_{n\downarrow}\rangle\right) \\ &= V_e \sum_n u_n v_n^* (1 - 2f_n), \quad (17)\end{aligned}$$

where $f_n = 1/(e^{\beta E_n}+1)$, $\beta \equiv 1/k_B T$. It has to be solved self-consistently in conjunction with Eq. (14).

Finally, to evaluate a one-body observable, one needs

$$\begin{aligned}\langle \hat{A}\rangle(\mathbf{r}) &= \sum_s \langle \psi_s^\dagger \hat{A} \psi_s\rangle \\ &= \sum_n \left\langle \left(u_n^* \gamma_{n\uparrow}^\dagger - v_n \gamma_{n\downarrow}\right) \hat{A} \left(u_n \gamma_{n\uparrow} - v_n^* \gamma_{n\downarrow}^\dagger\right)\right\rangle \\ &+ \sum_n \left\langle \left(v_n \gamma_{n\uparrow} + u_n^* \gamma_{n\downarrow}^\dagger\right) \hat{A} \left(v_n^* \gamma_{n\uparrow}^\dagger + u_n \gamma_{n\downarrow}\right)\right\rangle \\ &= 2\sum_n \left[u_n^*(\mathbf{r}) \hat{A} u_n(\mathbf{r}) f_n + v_n(\mathbf{r}) \hat{A} v_n^*(\mathbf{r})(1 - f_n)\right] \quad (18)\end{aligned}$$

C. Current-carrying state

The superconductor condensate is moving after a **Galilean boost**,

$$\psi_s(\mathbf{r}) \to e^{i\mathbf{q}\cdot\mathbf{r}} \psi_s(\mathbf{r}). \quad (19)$$

For a uniform system in equilibrium,

$$\psi_\uparrow(\mathbf{r}) = \sum_k \left[u_k(\mathbf{r})\gamma_{k\uparrow} - v_k^*(\mathbf{r})\gamma_{k\downarrow}^\dagger\right]. \quad (20)$$

Therefore one needs to choose

$$u_k(\mathbf{r}) = u_k e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}},$$
$$v_k(\mathbf{r}) = v_k e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}}. \quad (21)$$

Also,

$$\Delta(\mathbf{r}) = \Delta e^{2i\mathbf{q}\cdot\mathbf{r}}. \quad (22)$$

The BdG equation (for $U(\mathbf{r})$=0) is

$$\begin{pmatrix} \varepsilon_{k+q} & \Delta \\ \Delta & -\varepsilon_{k-q} \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = E_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}, \quad (23)$$

which has the eigenvalues,

$$E_k = \frac{\varepsilon_{k+q} - \varepsilon_{k-q}}{2} + \left[\left(\frac{\varepsilon_{k+q} + \varepsilon_{k-q}}{2}\right)^2 + \Delta^2\right]^{1/2}. \quad (24)$$

For $q \simeq \Delta/\hbar v_F \ll \varepsilon_F$,

$$E_k \simeq E_k^0 + \frac{\hbar^2}{m}\mathbf{k}\cdot\mathbf{q}, \quad (25)$$

in which E_k^0 is the excitation energy without current. The excitation energy for the current-carrying state could drop to zero when the current reaches a critical velocity

$$\frac{\hbar q_c}{m} = \frac{\Delta}{p_F} \equiv v_c. \quad (26)$$

Notice that if Δ itself depends on q, then its value needs to be solved self-consistently using

$$1 = V_e \sum_k \frac{1 - f_{k+q} - f_{k-q}}{2E_k}. \quad (27)$$

It's possible that when the current destroys E_k, Δ may still be finite. In this case, we have a gapless superconductor (Zagoskin, p.182).

The eigenstates are

$$\begin{aligned} u_k^2 &= \frac{1}{2} + \frac{1}{2}\frac{\frac{\varepsilon_{k+q}+\varepsilon_{k-q}}{2}}{\sqrt{\left(\frac{\varepsilon_{k+q}+\varepsilon_{k-q}}{2}\right)^2 + \Delta^2}} \simeq \frac{1}{2}\left(1 + \frac{\tilde{\varepsilon}_k}{E_k^0}\right) \\ v_k^2 &\simeq \frac{1}{2}\left(1 - \frac{\tilde{\varepsilon}_k}{E_k^0}\right), \quad (28)\end{aligned}$$

where $\tilde{\varepsilon}_k \equiv \varepsilon_k + \mathbf{v}\cdot\hbar\mathbf{k}$. From this we can calculate the current density with Eq. (18). First, we need

$$u_k^*(\mathbf{r})\frac{\mathbf{p}}{m}u_k(\mathbf{r}) = \frac{\hbar(\mathbf{k}+\mathbf{q})}{m}|u_k|^2,$$
$$v_k(\mathbf{r})\frac{\mathbf{p}}{m}v_k^*(\mathbf{r}) = -\frac{\hbar(\mathbf{k}-\mathbf{q})}{m}|v_k|^2. \quad (29)$$

The current density then follows as

$$\begin{aligned}\mathbf{j} &= 2e\sum_k \left[u_k^*(\mathbf{r})\frac{\mathbf{p}}{m}u_k(\mathbf{r})f_k + v_k(\mathbf{r})\frac{\mathbf{p}}{m}v_k^*(\mathbf{r})(1 - f_k)\right] \\ &\simeq 2\frac{e\hbar}{m}\sum_k \left[(\mathbf{k}+\mathbf{q})|u_k|^2 f_k - (\mathbf{k}-\mathbf{q})|v_k|^2(1 - f_k)\right] \quad (30)\end{aligned}$$

in which $E_k \simeq E_k^0 + \mathbf{v}\cdot\hbar\mathbf{k}$ inside the distribution function.

In general, for a non-uniform but slowly-varying current distribution, one can write (see p. 384 of Ref. 3)

$$\Delta(\mathbf{r}) = \Delta e^{i\theta(\mathbf{r})}$$
$$u_k(\mathbf{r}) = u_k e^{i\mathbf{k}\cdot\mathbf{r} + i\theta(\mathbf{r})/2}$$
$$v_k(\mathbf{r}) = v_k e^{i\mathbf{k}\cdot\mathbf{r} - i\theta(\mathbf{r})/2}. \quad (31)$$

D. Gauge transformation

Recall that the Schrodinger equation,

$$\frac{1}{2m}\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)^2\psi = E\psi, \tag{32}$$

is invariant under the following gauge transformation,

$$\begin{aligned}\mathbf{A}\to\mathbf{A}' &= \mathbf{A}+\nabla\chi,\\ \psi\to\psi' &= e^{-i\frac{e}{\hbar c}\chi(\mathbf{r})}\psi.\end{aligned} \tag{33}$$

Similarly, the BdG equation in Eq. (14) is invariant under

$$\begin{aligned}\mathbf{A}\to\mathbf{A}' &= \mathbf{A}+\nabla\chi,\\ \begin{pmatrix}u\\ v\end{pmatrix}\to\begin{pmatrix}u'\\ v'\end{pmatrix} &= \begin{pmatrix}e^{-i\frac{e}{\hbar c}\chi(\mathbf{r})}u\\ e^{+i\frac{e}{\hbar c}\chi(\mathbf{r})}v\end{pmatrix}\\ \Delta\to\Delta' &= e^{-2i\frac{e}{\hbar c}\chi(\mathbf{r})}\Delta.\end{aligned} \tag{34}$$

The pair potential Δ needs to remain single-valued before and after the gauge transformation. For a superconductor ring, this requires $(2e/\hbar c)(\chi(2\pi)-\chi(0))=2\pi m$, where m is an integer.

It is possible to remove the phase factor of $\Delta(\mathbf{r})$ in Eq. (31) after a gauge transformation. One can choose

$$\chi(\mathbf{r}) = -\frac{\hbar c}{2e}\theta(\mathbf{r}). \tag{35}$$

This also removes the phase $\theta(\mathbf{r})$ in $u_k(\mathbf{r})$ and $v_k(\mathbf{r})$. Notice that before the gauge transformation, the probability amplitudes $u_k(\mathbf{r})$ and $v_k(\mathbf{r})$ are single-valued. After the gauge transformation, however, the probability amplitudes are multiplied by $(-1)^n$ after one circle, where n is the flux quanta in the ring (see p. 152 of Ref. 1).

II. STRUCTURE OF A VORTEX

Consider a vortex line along the z-axis. Its structure is determined by the BdG equation. Choose the cylindrical coordinate and write

$$\begin{aligned}\Delta(\mathbf{r}) &= |\Delta(r)|e^{-i\phi}\\ u(\mathbf{r}) &= e^{i(\mu-1/2)\phi}e^{ik_z z}u_{n\mu k_z}(r)\\ v(\mathbf{r}) &= e^{i(\mu+1/2)\phi}e^{ik_z z}v_{n\mu k_z}(r),\end{aligned} \tag{36}$$

in which n is the quantum number for the radial direction. In order for $u(\mathbf{r})$ and $v(\mathbf{r})$ to be single-valued around the azimuthal angle, the angular momentum quantum number μ has to be an half-integer.

Substitute $u(\mathbf{r})$ and $v(\mathbf{r})$ into the BdG equation, and choose $\mathbf{A}=A_\phi\hat{\phi}$, so that

$$\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)^2 = -\hbar^2\frac{\partial^2}{\partial r^2}-\frac{\hbar^2}{r}\frac{\partial}{\partial r}+\left(\frac{\hbar}{ir}\frac{\partial}{\partial\phi}+\frac{e}{c}A_\phi\right)^2-\hbar^2\frac{\partial^2}{\partial z^2}, \tag{37}$$

we will get

$$\begin{aligned}-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)u_{n\mu k_z}+\frac{\hbar^2}{2m}\left(\frac{\mu-1/2}{r}+\frac{e}{\hbar c}A_\phi\right)^2u_{n\mu k_z}+\left(\frac{\hbar^2k_z^2}{2m}-\varepsilon_F\right)u_{n\mu k_z}+|\Delta|v_{n\mu k_z} &= E_{n\mu k_z}u_{n\mu k_z},\\ -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)v_{n\mu k_z}+\frac{\hbar^2}{2m}\left(\frac{\mu+1/2}{r}-\frac{e}{\hbar c}A_\phi\right)^2v_{n\mu k_z}+\left(\frac{\hbar^2k_z^2}{2m}-\varepsilon_F\right)v_{n\mu k_z}-|\Delta|u_{n\mu k_z} &= -E_{n\mu k_z}v_{n\mu k_z}.\end{aligned} \tag{38}$$

This has to be solved in conjunction with

$$\Delta(\mathbf{r}) = V_e\sum_{E_{n\mu k_z}<\hbar\omega_c}u_{n\mu k_z}(r)v^*_{n\mu k_z}(r)\left[1-2f(E_{n\mu k_z})\right]. \tag{39}$$

Some nice numerical results of this calculation can be found in Ref. 4.

After the coupled equations are solved, the eigenvalues $E_{n\mu k_z}$ could give the energy of the bound states inside the vortex, and the eigenstates $(u_{n\mu k_z}, v_{n\mu k_z})$ determine the profile of $\Delta(r)$. An often used simplification is to adopt an approximate but acceptable form of $\Delta(r)$, such as

$$\Delta(r) = \begin{cases}\Delta_0\frac{r}{\xi} & \text{for } r<\xi_c\\ \Delta_0 & \text{for } r>\xi_c\end{cases} \tag{40}$$

The parameter ξ_c can be determined self-consistently.

It is possible to solve approximately, but analytically, the bound state energies deep inside the core of the vortex. For example, on p. 386 of Ref. 3, one has (for the lowest radial mode n, $k_z=0$),

$$E_\mu \simeq \mu\times\frac{\int_0^\infty\frac{\Delta(\rho)}{\rho}e^{-2K(\rho)}d\rho}{\int_0^\infty e^{-2K(\rho)}d\rho}, \tag{41}$$

in which $\rho\equiv k_\perp r=\sqrt{k_F^2-k_z^2}\,r$, and $K(\rho)=\frac{m}{\hbar^2k_\perp^2}\int_0^\rho d\rho'\Delta(\rho')$. That is, the bound state energies are proportional to the half integer μ (see Fig. 1)). Recall that it is a half integer because of the boundary condition around the azimuthal angle ϕ.

In certain systems, an electron spin would couple with

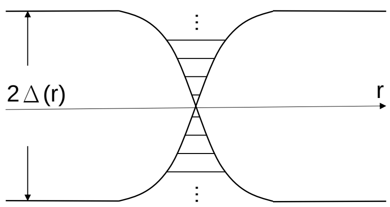

FIG. 1 The bound states within a vortex core.

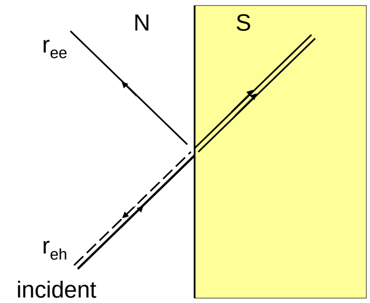

FIG. 2 An electron incident on a $N-S$ interface can be reflected either as an electron or as a hole. The charge can flow through the superconductor as a moving Cooper pair.

its own orbital motion, so that when the electron circles once, the spin rotates by 2π. This adds a Berry phase of π to the wave function. If this happens, then the wave function changes sign after circling around the vortex once. As a result, the bound state energy would locate at integer values of μ. In particular, a single bound state at $\mu=0$ cannot be easily perturbed away from zero energy since it is protected by the particle-hole symmetry of a superconductor state. Such a **Majorana fermion** state can be found in a Rashba system in proximity with a superconductor.

Once $u_\alpha(r)$ and $v_\alpha(r)$ are solved ($\alpha=n,\mu,k_z$), one can calculate things like the distribution of current density, the **local density of states** (LDOS)... etc around the vortex. For example, the current density is given by

$$\begin{aligned} J_\phi(r) &= \frac{2e}{m}\sum_\alpha\left[f_\alpha|u_\alpha|^2\left(\hbar\frac{\mu-1/2}{r}+\frac{e}{c}A_\phi\right)\right. \quad (42) \\ &+ \left.(1-f_\alpha)|v_\alpha|^2\left(-\hbar\frac{\mu+1/2}{r}+\frac{e}{c}A_\phi\right)\right]. \end{aligned}$$

The local density of states is given by

$$\begin{aligned} A(\mathbf{r},\omega) &= \frac{1}{2}\sum_{\alpha s}\left[\left|\langle\Psi_\alpha|\psi_s^\dagger(\mathbf{r})|\Psi_0\right|^2\delta(\omega-\omega_{\alpha0})\right. \quad (43) \\ &+ \left.\left|\langle\tilde{\Psi}_\alpha|\psi_s(\mathbf{r})|\Psi_0\right|^2\delta(\omega+\omega_{\alpha0})\right]. \end{aligned}$$

The manybody state Ψ_α ($\tilde{\Psi}_\alpha$) has one more (less) particle than the state Ψ_0, and $\omega_{\alpha0}$ is the energy difference between state-α and state-0.

In a superconductor, the field operators are related to the bogolon operators as follows (see Eq. (8)),

$$\begin{aligned} \psi_\uparrow &= \sum_\alpha\left(u_\alpha\gamma_{\alpha\uparrow}-v_\alpha^*\gamma_{\alpha\downarrow}^\dagger\right), \quad (44) \\ \psi_\downarrow &= \sum_\alpha\left(v_\alpha^*\gamma_{\alpha\uparrow}^\dagger+u_\alpha^*\gamma_{\alpha\downarrow}\right). \end{aligned}$$

A direct calculation gives

$$\begin{aligned} \sum_s\left|\langle\Psi_\alpha|\psi_s^\dagger(\mathbf{r})|\Psi_0\right|^2 &= 2|u_\alpha|^2, \quad (45) \\ \sum_s\left|\langle\tilde{\Psi}_\alpha|\psi_s(\mathbf{r})|\Psi_0\right|^2 &= 2|v_\alpha|^2. \end{aligned}$$

Therefore,

$$A(\mathbf{r},\omega)=\sum_\alpha|u_\alpha|^2\delta(\omega-\omega_{\alpha0})+|v_\alpha|^2\delta(\omega+\omega_{\alpha0}), \quad (46)$$

where $\omega_{\alpha0}=E_\alpha$. That is, adding an electron to state-α gives a positive energy peak at E_α, with a weight $|u_\alpha|^2$; removing an electron from state-α gives a negative energy peak at $-E_\alpha$, with a weight $|v_\alpha|^2$. The LDOS of a vortex can be probed experimentally by **Scanning Tunnelling Microscope** (STM) (e.g., Hess et al, PRL 1990).

III. ANDREEV REFLECTION

An electron impinging on an interface between two different materials could either be reflected or transmitted. In 1964, while trying to explain an extra thermal resistance in a Normal metal -Superconductor (NS) junction, Andreev discovered a peculiar process: an electron impinging on a NS interface from the N-side could be reflected as a hole, as shown in Fig. 2. At the mean time, a Cooper pair with two electrons emerges on the S-side and keep the charge flow going. Notice that the hole is reflected back to the impinging path, similar to a phase-conjugate mirror in nonlinear optics.

In the following, we will calculate the amplitudes of reflection and transmission using the BdG equation. Firstly, assume the pairing potential is a step function, $\Delta(\mathbf{r})=\Delta_0\theta(x)$. In reality, the magnitude of $\Delta(\mathbf{r})$ would decrease near the boundary of the superconductor. Such a change is neglected here. The BdG equation on both sides of the interface is

$$\begin{pmatrix}H_0 & \Delta\\ \Delta & -H_0\end{pmatrix}\begin{pmatrix}u_n\\ v_n\end{pmatrix}=E_n\begin{pmatrix}u_n\\ v_n\end{pmatrix}. \quad (47)$$

Since the system is uniform along the interface, we can choose

$$\begin{aligned} u_n(\mathbf{r}) &= u(x)e^{ik_y y+ik_z z}, \\ v_n(\mathbf{r}) &= v(x)e^{ik_y y+ik_z z}. \end{aligned} \quad (48)$$

It is not difficult to see that, the BdG equation for $\psi(x) = (u(x), v(x))^T$ are similar to the one above. One only needs to replace the μ in H_0 with $\mu' = \mu - (k_y^2 + k_z^2)/2m$.

On the N-side ($x < 0$) with $\Delta(\mathbf{r}) = 0$, $u(x)$ and $v(x)$ decouple with each other, so that the electron travelling wave and hole travelling wave are

$$\begin{aligned} \psi_e^{\pm}(x) &= e^{\pm ik_+ x}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \psi_h^{\pm}(x) &= e^{\pm ik_- x}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \quad (49)$$

They are accompanied with different dynamic phases: $e^{-iEt/\hbar}$ for electron, and $e^{iEt/\hbar}$ for hole, so ψ_h^+ is actually moving to the left, not to the right. The momenta $k_\pm$ are defined via

$$\pm E = \frac{\hbar^2}{2m}\left(k_\pm^2 + k_y^2 + k_z^2\right) - \mu = \frac{\hbar^2 k_\pm^2}{2m} - \mu'. \quad (50)$$

That is,

$$k_\pm = \sqrt{2m}\sqrt{\mu' \pm E}. \quad (51)$$

On the S-side ($x > 0$), $u(x)$ and $v(x)$ couple with each other and the eigenstates are bogolon states,

$$\begin{aligned} \psi_{eb}^{\pm}(x) &= e^{\pm iq_+ x}\begin{pmatrix} u \\ v \end{pmatrix}, \\ \psi_{hb}^{\pm}(x) &= e^{\pm iq_- x}\begin{pmatrix} -v \\ u \end{pmatrix}. \end{aligned} \quad (52)$$

The momenta $q_\pm$ are defined via

$$\pm\sqrt{E^2 - \Delta_0^2} = \frac{\hbar^2 q_\pm^2}{2m} - \mu', \quad (53)$$

so that

$$q_\pm = \sqrt{2m}\sqrt{\mu' \pm \sqrt{E^2 - \Delta_0^2}}. \quad (54)$$

The travelling wave would become evanescent wave if $E < \Delta_0$.

For an electron impinging from the left, one has

$$\begin{aligned} \psi_L(x) &= \psi_e^+ + r_{ee}\psi_e^- + r_{eh}\psi_h^+, \\ \psi_R(x) &= t_{ee}\psi_{eb}^+ + t_{eh}\psi_{hb}^-. \end{aligned} \quad (55)$$

The boundary condition is

$$\begin{aligned} \psi_L(0) &= \psi_R(0), \\ \psi_L'(0) &= \psi_R'(0). \end{aligned} \quad (56)$$

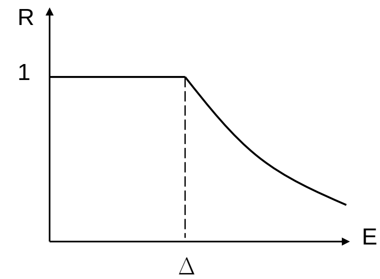

FIG. 3 The hole reflectivity as a function of energy. For a transparent interface, the incident electron is always reflected as a hole when $E < \Delta$

After a lengthy calculation, one can get

$$r_{eh} = \frac{2}{\frac{u}{v}\frac{q_- + k_-}{q_+ + q_-}\left(1 + \frac{q_+}{k_+}\right) + \frac{v}{u}\frac{q_+ - k_-}{q_+ + q_-}\left(1 - \frac{q_-}{k_+}\right)}. \quad (57)$$

This is the probability amplitude for an electron to turn into a hole upon reflection. The other coefficients r_{ee}, t_{ee}, and t_{eh} can also be obtained accordingly (see p. 188 of Ref. 2).

Finally, we still need to solve for the amplitudes (u, v). Substitute

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = e^{\pm iq_+ x}\begin{pmatrix} u \\ v \end{pmatrix} \quad (58)$$

to the BdG equation. One would get the familiar expressions,

$$\begin{aligned} u^2 &= \frac{1}{2}\left(1 + \frac{\varepsilon}{E}\right) = \frac{1}{2}\left(1 + \sqrt{1 - \frac{\Delta^2}{E^2}}\right), \\ v^2 &= \frac{1}{2}\left(1 - \frac{\varepsilon}{E}\right) = \frac{1}{2}\left(1 - \sqrt{1 - \frac{\Delta^2}{E^2}}\right). \end{aligned} \quad (59)$$

For $\mu' \gg E$ and Δ_0, $k_\pm \simeq q_\pm \simeq \sqrt{2m\mu'} \equiv k_F'$, and

$$k_+ - q_- \simeq q_+ - k_- \simeq k_F'\frac{u}{v}\frac{\Delta_0}{2\mu'}. \quad (60)$$

Therefore,

$$r_{eh} \simeq \frac{1}{\frac{u}{v}\left[1 + \left(\frac{\Delta_0}{2\mu'}\right)^2\right]} \simeq \frac{v}{u}, \quad (61)$$

and the reflectivity

$$R \equiv |r_{eh}|^2 = \begin{cases} \frac{E - \sqrt{E^2 - \Delta_0^2}}{E + \sqrt{E^2 - \Delta_0^2}} & \text{for } E > \Delta_0 \\ 1 & \text{for } E \le \Delta_0 \end{cases} \quad (62)$$

When $E < \Delta_0$, the electron would always turn into a hole upon reflection (see Fig. 3). However, this is true

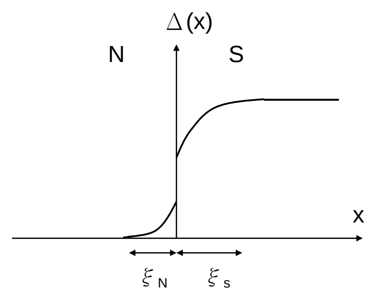

FIG. 4 The pairing potential $\Delta(x)$ near a $N-S$ interface.

only for a transparent interface. If one adds an additional barrier $Z\delta(x)$ at the interface, then the electron has some probability amplitude r_{ee} to be reflected as an electron via the usual specular reflection. More details can be found in Ref. 5.

As we mentioned earlier, in reality, $\Delta(\mathbf{r})$ would not be a step function near the interface. Its actual form can be obtained be solving the BdG equation and the pairing potential together. For a schematic plot of $\Delta(x)$, see Fig. 4. Notice that the pairing potential spills over into the normal metal, so that the metallic electron near the interface would also be superconducting. This is called the **proximity effect**. The length scales of ξ_N and ξ_S are roughly the same as the **coherence length** of the superconductor. It is of the order of 1 μm for type-I superconductors.

IV. QUANTUM TUNNELLING

We now consider a SIS structure, in which a thin layer of insulator is sandwiched between two superconductors. If the insulator is thin enough, then the condensate wave function from two sides would overlap with each other. This could cause two types of quantum tunnelling: single-particle tunnelling and Cooper-pair tunnelling. The latter, usually called the **Josephson tunnelling**, is a coherent process that depends on the phase difference of the two superconductor condensates.

We will start from the basics of perturbation theory. First divide the Hamiltonian into three parts,

$$H = H_L + H_R + H_T. \tag{63}$$

The superconductor on the left (H_L) couples with the one on the right (H_R) because of H_T. The tunnelling Hamiltonian is of the form,

$$H_T = \sum_{pqs} \left(t_{pq} a^\dagger_{ps} b_{qs} + t^*_{pq} b^\dagger_{qs} a_{ps} \right), \tag{64}$$

in which $a^\dagger$ ($b^\dagger$) is the creation operator for the left (right) superconductor. The electron spin would not flip during the tunnelling.

The particle number operator for the left part is

$$N_L = \sum_{ps} a^\dagger_{ps} a_{ps}. \tag{65}$$

The current operator is

$$\begin{aligned} I &= (-e)\dot{N}_L \\ &= -e\frac{i}{\hbar}[H, N_L] = -e\frac{i}{\hbar}[H_T, N_L], \end{aligned} \tag{66}$$

where we have assumed (as others did) that $[H_L, N_L] = 0$. Carry out the commutator, one will get

$$I = i\frac{e}{\hbar}\sum_{pqs}\left(t_{ps} a^\dagger_{ps} b_{qs} - t^*_{pq} b^\dagger_{qs} a_{ps} \right). \tag{67}$$

Its thermal average is

$$\langle I \rangle = \sum_n P_n \langle \Psi_n | I | \Psi_n \rangle, \; P_n = \frac{e^{-\beta E_n}}{Z}, \tag{68}$$

in which Ψ_n and E_n are the eigenstates and eigenenergies for *the whole system*.

In the absence of tunnelling,

$$(H_L + H_R)|\Psi^0_n\rangle = E^0_n |\Psi^0_n\rangle, \tag{69}$$

where $|\Psi^0_n\rangle = |L^0\rangle \otimes |R^0\rangle$, and $E^0_n = E^0_L + E^0_R$. The tunnelling part is treated as a perturbation, and

$$|\Psi_n\rangle \simeq |\Psi^0_n\rangle + \sum_{m\neq n} \frac{|\Psi^0_m\rangle\langle\Psi^0_m|H_T|\Psi^0_n\rangle}{E^0_n - E^0_m + i\delta}. \tag{70}$$

We have added an infinitesimal $i\delta$ to prevent the divergence due to a continuous spectrum.

Since there can be no current in the absence of tunnelling, the zeroth order current is zero,

$$\langle \Psi^0_n | T | \Psi^0_n \rangle = 0. \tag{71}$$

The first order current is

$$\begin{aligned} \langle I \rangle &= \sum_n P_n (\langle \Psi^0_n | I | \Psi^1_n \rangle + \langle \Psi^1_n | I | \Psi^0_n \rangle) \\ &= \sum_{nm} P_n \left(\frac{\langle \Psi^0_n | I | \Psi^0_m \rangle \langle \Psi^0_m | I | \Psi^0_n \rangle}{E^0_n - E^0_m + i\delta} + h.c. \right). \end{aligned} \tag{72}$$

Substitute the H_T in Eq. (64) to the equation above. After some tedious rearrangement, we have (the superscript 0 is now omitted for brevity)

$$\begin{aligned} \langle I \rangle &= i\frac{e}{\hbar} \sum_{nm}\sum_{pqs}\sum_{p'q's'} P_n \\ &\times \Big[t_{pq} t_{p'q'} \langle \Psi_n | a^\dagger_{ps} b_{qs} | \Psi_m \rangle \langle \Psi_m | a^\dagger_{p's'} b_{q's'} | \Psi_n \rangle D^+_{nm} \\ &- t^*_{pq} t^*_{p'q'} \langle \Psi_n | b^\dagger_{qs} a_{ps} | \Psi_m \rangle \langle \Psi_m | b^\dagger_{q's'} a_{p's'} | \Psi_n \rangle D^+_{nm} \\ &+ t_{pq} t^*_{p'q'} \langle \Psi_n | a^\dagger_{ps} b_{qs} | \Psi_m \rangle \langle \Psi_m | b^\dagger_{q's'} a_{p's'} | \Psi_n \rangle D^-_{nm} \\ &- t^*_{pq} t_{p'q'} \langle \Psi_n | b^\dagger_{qs} a_{ps} | \Psi_m \rangle \langle \Psi_m | a^\dagger_{p's'} b_{q's'} | \Psi_n \rangle D^-_{nm} \Big], \end{aligned} \tag{73}$$

in which

$$D^{\pm}_{nm} \equiv \frac{1}{E_n - E_m + i\delta} \pm \frac{1}{E_n - E_m - i\delta}. \qquad (74)$$

One can get some feeling of particle motion from the expression above. Recall that operator-a (-b) is for particles to the left (right) of the insulator. For the third line within the square bracket, start from $|\Psi_n\rangle$ on the right end, an electron first moves from left to right ($b^\dagger_{q's'}a_{p's'}$), and then from right to left ($a^\dagger_{ps}b_{qs}$). The fourth line has a similar tunnelling process, but the direction is reversed. These two lines account for the single-particle tunnelling (designated as $\langle I_2\rangle$).

On the other hand, for the first (second) line, in each bracket an electron jump from right to left (left to right). So they represent Cooper-pair tunnelling (designated as $\langle I_1\rangle$).

A. Single-particle tunnelling

We will start from the single-particle tunnelling. Notice that

$$D^-_{nm} = -2\pi i\delta(E_n - E_m), \qquad (75)$$

so that the last two lines of Eq. (73) give

$$\langle I_2\rangle = 2\pi\frac{e}{\hbar}\sum_{nm}\sum_{pqs} P_n|t_{pq}|^2 \qquad (76)$$
$$\times \left[|\langle\Psi_m|b^\dagger_{qs}a_{ps}|\Psi_n\rangle|^2 - |\langle\Psi_m|a^\dagger_{ps}b_{qs}|\Psi_n\rangle|^2\right]\delta(E_n - E_m).$$

Unperturbed $|\Psi_n\rangle$ and $|\Psi_m\rangle$ are direct product of L and R states, therefore

$$|\langle\Psi_m|b^\dagger_{qs}a_{ps}|\Psi_n\rangle|^2 = |\langle R'|b^\dagger_{qs}|R\rangle|^2|\langle L'|a_{ps}|L\rangle|^2. \qquad (77)$$

Also,

$$\begin{aligned} E_n - E_m &= E_L + E_R - (E_{L'} + E_{R'}), \qquad (78)\\ \delta(E_n - E_m) &= \int d\omega\,\delta(\omega \pm (E_{L'} - E_L))\delta(\omega \mp (E_{R'} - E_R)),\\ \text{and}\quad P_n &= \frac{e^{-\beta E_L}}{Z_L}\frac{e^{-\beta E_R}}{Z_R} = P_L P_R. \end{aligned}$$

Define

$$\begin{aligned} A^{\gtrless}_{\omega ps} &= \sum_{LL'} P_L|\langle L'|a_{ps}|L\rangle|^2\delta(\omega \mp (E_{L'} - E_L)), \qquad (79)\\ A^{\gtrless}_{\omega qs} &= \sum_{RR'} P_R|\langle R'|b^\dagger_{qs}|R\rangle|^2\delta(\omega \mp (E_{R'} - E_R)). \end{aligned}$$

Then Eq. (76) can be simplified as

$$\langle I_2\rangle = \frac{2\pi e}{\hbar}\int d\omega\sum_{pqs}|t_{pq}|^2\left(A^<_{\omega ps}A^>_{\omega qs} - A^>_{\omega ps}A^<_{\omega qs}\right). \qquad (80)$$

For non-interacting particles,

$$\langle L'|a_{ps}|L\rangle \neq 0 \;\text{ only if }\; |L'\rangle = a_{ps}|L\rangle. \qquad (81)$$

Therefore,

$$\begin{aligned} A^<_{\omega ps} &= \sum_L P_L\langle L|\hat{n}_{ps}|L\rangle\delta(\omega - \varepsilon_p) \qquad (82)\\ &= f_p\delta(\omega - \varepsilon_p). \end{aligned}$$

Similarly,

$$A^>_{\omega qs} = (1 - f_q)\delta(\omega - \varepsilon_q). \qquad (83)$$

Finally,

$$\langle I_2\rangle = \frac{2\pi e}{\hbar}\sum_{pqs}|t_{pq}|^2\left[f_p(1 - f_q) - f_q(1 - f_p)\right]\delta(\varepsilon_p - \varepsilon_q). \qquad (84)$$

This non-interacting result can also be derived by the **Fermi golden rule**. A current flows only if there is an unbalance in the distribution functions, f_p and f_q, which is usually caused by an external bias.

In order to evaluate $A^{>,<}$ for the condensate state, it is more convenient to write the a and b operators in terms of bogolon operators. For example,

$$\begin{aligned} a_{p\uparrow} &= u_p\gamma_{p\uparrow} + v_p\gamma^\dagger_{-p\downarrow}, \qquad (85)\\ a^\dagger_{-p\downarrow} &= -v^*_p\gamma_{p\uparrow} + u_p\gamma^\dagger_{-p\downarrow}. \end{aligned}$$

It is left as an exercise to show that

$$A^<_{\omega p\uparrow} = u^2_p f_p\delta(\omega - \varepsilon_p) + |v_p|^2(1 - f_p)\delta(\omega + \varepsilon_p) = A^<_{\omega p\downarrow}, \qquad (86)$$

where ε_p is the bogolon energy, and

$$A^>_{\omega q\uparrow} = u^2_q(1 - f_q)\delta(\omega - \varepsilon_q) + |v_q|^2 f_q\delta(\omega + \varepsilon_q) = A^>_{\omega q\downarrow}. \qquad (87)$$

Their sum is the usual **spectral function**,

$$\begin{aligned} A_{\omega ps} &= A^>_{\omega ps} + A^<_{\omega ps} \qquad (88)\\ &= u^2_p\delta(\omega - \varepsilon_p) + |v_p|^2\delta(\omega + \varepsilon_p). \end{aligned}$$

Notice the similarity with the LDOS in Eq. (43). If $\Delta = 0$, then these bogolons become the usual electron and hole,

$$A_{\omega ps} = \delta(\omega - \varepsilon_p) + \delta(\omega + \varepsilon_p). \qquad (89)$$

B. Cooper-pair tunnelling

We now calculate the first two lines within the square bracket of Eq. (73). According to Eq. (74),

$$D^+_{nm} = 2\mathcal{P}\frac{1}{E_n - E_m}, \qquad (90)$$

where $\mathcal{P}\frac{1}{x}$ takes the principle value of $\frac{1}{x}$. Notice that since there is no delta function (as in D^-_{nm}), the intermediate states $|\Psi_m\rangle$ are allowed to have energies different from E_n.

Rewriting the electron operators as bogolon operators, we have

$$\begin{aligned} & \sum_{pqs} t_{pq} a^\dagger_{ps} b_{qs} \qquad (91)\\ = & \sum_{pq} \left(t_{pq} a^\dagger_{p\uparrow} b_{q\uparrow} + t^*_{pq} a^\dagger_{-p\downarrow} b_{-q\downarrow} \right), \quad t_{-p-q} = t^*_{pq} \\ = & \sum_{pq} t_{pq} (\underbrace{u_p u_q \gamma^\dagger_p \gamma_q}_{1} + \underbrace{u_p v_q \gamma^\dagger_p \gamma^\dagger_{-q}}_{2} + \underbrace{v^*_p u_q \gamma_{-p} \gamma_q}_{3} + \underbrace{v^*_p v_q \gamma_{-p} \gamma^\dagger_{-q}}_{4}) \\ + & \; t^*_{pq} (\underbrace{v^*_p v_q \gamma_p \gamma^\dagger_q}_{5} - \underbrace{v^*_p u_q \gamma_p \gamma_{-q}}_{6} - \underbrace{u_p v_q \gamma^\dagger_{-p} \gamma^\dagger_q}_{7} + \underbrace{u_p u_q \gamma^\dagger_{-p} \gamma_{-q}}_{8}). \end{aligned}$$

It is not difficult to see that, in the product

$$\langle \Psi_n | \sum_{pqs} t_{pq} a^\dagger_{ps} b_{qs} | \Psi_m \rangle \langle \Psi_m | \sum_{p'q's'} t_{p'q'} a^\dagger_{p's'} b_{q's'} | \Psi_n \rangle, \qquad (92)$$

the only non-zero terms are $1\times5'$, $2\times6'$, $3\times7'$, and $4\times8'$. For example,

$$\begin{aligned} & \sum_m \langle \Psi_n | \gamma^\dagger_p \gamma_q | \Psi_m \rangle \langle \Psi_m | \gamma_{p'} \gamma^\dagger_{q'} | \Psi_n \rangle \qquad (93)\\ = & \; \langle \Psi_n | \gamma^\dagger_p \gamma_q \gamma_{p'} \gamma^\dagger_{q'} | \Psi_n \rangle, \quad (E_m - E_n = \varepsilon_q - \varepsilon_p) \\ = & \; \langle L | \gamma^\dagger_p \gamma_{p'} | L \rangle \langle R | \gamma_q \gamma^\dagger_{q'} | R \rangle \\ \rightarrow & \; f_p (1 - f_q) \delta_{pp'} \delta_{qq'} \quad \text{after} \quad \sum_n P_n . \end{aligned}$$

Let the phases of the two condensates be different,

$$\begin{aligned} v_p &= |v_p| e^{i\theta_1}, \qquad (94)\\ v_q &= |v_q| e^{i\theta_2}. \end{aligned}$$

After some calculations, the Josephson current is found to be

$$\begin{aligned} \langle I_1 \rangle &= i \frac{e}{\hbar} \sum_{nm} P_n \langle \Psi_n | \sum_{pqs} t_{pq} a^\dagger_{ps} b_{qs} | \Psi_m \rangle \langle \Psi_m | \sum_{p'q's'} t_{p'q'} a^\dagger_{p's'} b_{q's'} | \Psi_n \rangle 2 \mathcal{P} \frac{1}{E_n - E_m} - h.c. \qquad (95)\\ &= i \frac{e}{\hbar} \sum_{pq} |t_{pq}|^2 u_p u_q |v_p v_q| e^{-i(\theta_1 - \theta_2)} \\ &\times \; 2 \left[\frac{f_p (1 - f_q)}{\varepsilon_p - \varepsilon_q} + \frac{f_p f_q}{-\varepsilon_p - \varepsilon_q} + \frac{(1 - f_p)(1 - f_q)}{\varepsilon_p + \varepsilon_q} + \frac{(1 - f_p) f_q}{-\varepsilon_p + \varepsilon_q} \right] - h.c. \\ &= 4 \frac{e}{\hbar} \sum_{pq} |t_{pq}|^2 |u_p u_q v_p v_q| \sin(\theta_1 - \theta_2) \left(\frac{f_p - f_q}{\varepsilon_p - \varepsilon_q} + \frac{1 - f_p - f_q}{\varepsilon_p + \varepsilon_q} \right). \end{aligned}$$

Subsequently, we use

$$|u_p v_p| = \frac{\Delta_1}{2\varepsilon_p}, \qquad (96)$$

and

$$\begin{aligned} \sum_p &= V_1 \int \frac{d^3 p}{(2\pi)^3} = \int_{-\infty}^{\infty} d\varepsilon^0_p \frac{N_{n_1}(\varepsilon^0_p)}{2} \qquad (97)\\ &= \int_0^\infty d\varepsilon_p N_{s_1}(\varepsilon_p), \end{aligned}$$

where $N_n(\varepsilon^0)$ is the DOS for a normal metal, and

$$N_s(\varepsilon) = N_n(\varepsilon^0) \frac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta^2}} \theta(\varepsilon - \Delta). \qquad (98)$$

As a result, the Josephson current can be written as

$$\begin{aligned} \langle I_1 \rangle &= \frac{2e}{\hbar} \int_{\Delta_1}^{\infty} d\varepsilon_1 \int_{\Delta_2}^{\infty} d\varepsilon_2 N_{s_1}(\varepsilon_1) N_{s_2}(\varepsilon_2) \overline{|t^2|} \frac{\Delta_1 \Delta_2}{\varepsilon_1 \varepsilon_2} \qquad (99)\\ &\times \; \sin(\theta_1 - \theta_2) \left(\frac{f_1 - f_2}{\varepsilon_1 - \varepsilon_2} + \frac{1 - f_1 - f_2}{\varepsilon_1 + \varepsilon_2} \right), \end{aligned}$$

where

$$\overline{|t^2|} \equiv \int \frac{d\Omega_1}{4\pi} \int \frac{d\Omega_2}{4\pi} |t_{pq}|^2 \qquad (100)$$

is an average over the solid-angle of both Fermi surfaces.

After re-written the DOS N_{s_1}, N_{s_2} as N_{n_1}, N_{n_2}, the energy integrals can be evaluated by the method of contour integration. For details, one can see p.551 of Ref. 6. Here we only give the result: Define

$$G_n = \frac{e^2}{\hbar} N_{n_1} N_{n_2} \overline{|t^2|}, \qquad (101)$$

which is the conductance of a NIN junction. If $\Delta_1 = \Delta_2$, then the Josephson current is

$$\langle I_1 \rangle = I_c \sin(\theta_1 - \theta_2), \qquad (102)$$

where

$$I_c = \frac{\pi}{2e} G_n \Delta \tanh(\beta\Delta/2). \qquad (103)$$

This is called the **Ambegaokar-Baratoff formula**.

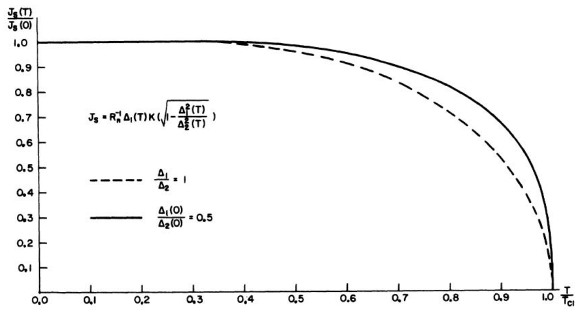

FIG. 5 Temperature dependence of the Josephson current. Dashed line is for $\Delta_1 = \Delta_2$; solid line is for $\Delta_1 = \Delta_2/2$. The figure is taken from Ambegaokar and Baratoff, Phys. Rev. Lett. **10**, 486 (1963).

If $\Delta_1 \neq \Delta_2$, then Anderson showed that, for $T \ll \Delta_{1,2}$,

$$I_c = \frac{2G_n}{e} \frac{\Delta_1 \Delta_2}{\Delta_1 + \Delta_2} K\left(\frac{|\Delta_1 - \Delta_2|}{\Delta_1 + \Delta_2}\right), \qquad (104)$$

where $K(x) (x \in [0, 1])$ is the elliptic integral of the first kind. $K(0) = \pi/2$, and it increases monotonically till $x = 1$, where it diverges.

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